IISER PUNE
MATHEMATICS COMPREHENSIVE EXAM (ALGEBRA)
SPRING 2014

All rings are commutative and contain $1 \neq 0$ *unless specified otherwise.*

(Q.1) (a) Show that the group $G = \{(a, b, c, d) \in \mathbb{Z}^4 \mid a+b+c+d = 0\}$ is isomorphic to $\mathbb{Z}^3$. (4)

(b) Let G be a finite simple group and p be a prime number which divides $|G|$. If G has exactly n Sylow p-subgroups ($n > 1$), then show that G is isomorphic to a subgroup of A_n. (6)

(Q.2) Let $n > 2$ be an integer and K be the cyclotomic extension of $\mathbb{Q}$ containing all n^{th} roots of unity.

(a) Show that the complex conjugation map $\tau : \mathbb{C} \to \mathbb{C}$ restricts to an field automorphism of K and gives a order two element in $\mathrm{Gal}(K : \mathbb{Q})$. (4)

(b) Find the degree of the fixed field $F := \{z \in K : \tau(z) = z\}$ over $\mathbb{Q}$ and find primitive element $\alpha \in K$ for F (which means $F = \mathbb{Q}(\alpha)$). (6)

(Q.3) Let A be a ring and $\mathcal{P}_1 \subsetneq \mathcal{P}_2 \subsetneq \mathcal{P}_3$ be a chain of prime ideals of the polynomial ring $A[X]$. Show that the following cannot be true.

$$\mathcal{P}_1 \cap A = \mathcal{P}_2 \cap A = \mathcal{P}_3 \cap A.$$

(*Hint*: By going modulo $\mathcal{P}_1 \cap A$, we can w.l.o.g. assume that A is an integral domain. Now consider its quotient field.) (10)

(Q.4) Let R be a commutative ring with $1 \neq 0$ and I, J be its ideals. Show that as R-modules,

$$R/I \otimes R/J \cong R/(I + J).$$

(10)

(Q.5) Let M be a finitely generated module over a ring R and $\Phi : M \to R^n$ be a surjective homomorphism. Show that the ker Φ is finitely generated submodule of M. (10)

(Q.6) Let R be an principal integral domain.

(a) Let F be the quotient field of R and M a finitely generated R-module. Then show that

$$\dim_F (F \otimes_R M) = \operatorname{rank}_R(M).$$

(5)

(b) Let m, n be positive integers such that

$$R^m \cong R^n$$

as R-modules. Show that $m = n$. (5)

(Q.7) (a) Let E_{ij} be the $n \times n$ matrix with 1 at $(i,j)^{th}$ place and 0 elsewhere. Prove that $SL(n, \mathbb{Z})$ is generated by the set $\{I_n + E_{ij} : 1 \leq i, j \leq n,\ i \neq j\}$, where I_n is $n \times n$ identity matrix. (8)

(b) Using (a), show that $SL(n, \mathbb{Z}) = [SL(n, \mathbb{Z}),\ SL(n, \mathbb{Z})]$. (2)

(*Hint* : For the first part, using Euclidean algorithm, try to convert the first row and column of the given matrix into standard basis vectors. Then use induction.)

Analysis Comprehensive Exam, December 23, 2013

1.(10 pts) Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function. Prove that the graph of f

$$G(f) = \{(x, y) \in \mathbf{R}^2 : x \in \mathbf{R}, y = f(x)\}$$

has Lebesgue measure 0.

2. Let $f : \mathbf{C} \to \mathbf{C}$ be a continuously differentiable function (in the real sense) and suppose that there exist constants $p, q > 1$ such that

$$|f(z)| \le \frac{1}{(1+|z|)^p},$$
$$\left|\frac{\partial f}{\partial x}(z)\right|, \left|\frac{\partial f}{\partial y}(z)\right| \le \frac{1}{(1+|z|)^q},$$

for all $z \in \mathbf{C}$.

(a) (5 pts) Prove that the integral

$$\int_{\mathbf{C}} \frac{f(\zeta)}{\zeta - z}\, dA(\zeta)$$

is convergent for any $z \in \mathbf{C}$. Here, dA is the Lebesgue measure on $\mathbf{C} \approx \mathbf{R}^2$. (**Hint:** It might be helpful to break up the integral into integrals over separate regions.)

(b) (5 pts) Justify why the following equalities are valid, i.e., why we can differentiate under the integral sign in the following manner:

$$\frac{\partial}{\partial \overline{z}} \int_{\mathbf{C}} \frac{f(\zeta)}{\zeta - z}\, dA(\zeta) = \frac{\partial}{\partial \overline{z}} \int_{\mathbf{C}} \frac{f(w+z)}{w} dA(w) = \int_{\mathbf{C}} \frac{1}{w}\frac{\partial f}{\partial \overline{w}}(z+w)\, dA(w).$$

3. Suppose both $f, \hat{f} \in L^1(\mathbf{R})$ and $\|f\|_1 < 1$.

(a) (3 pts) Prove that the series

$$F = f + f * f + f * f * f + \cdots$$

converges in $L^1(\mathbf{R})$.

(b) (3 pts) Prove that the integral

$$G(x) = \int_{-\infty}^{\infty} \frac{\hat{f}(t)}{1 - \hat{f}(t)}\, e^{ixt}\, dt$$

is covergent for any $x \in \mathbf{R}$.

(c) (4 pts) Prove that $F(x) = \alpha G(x)$ a.e. where α is some positive constant.

Remarks and instructions: Please read carefully

- Recall that for $f, g \in L^1(\mathbf{R})$,

$$\|f\|_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x)|\, dx$$
$$f * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)\, dy, \quad x \in \mathbf{R}$$
$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-ixt}\, dt, \quad x \in \mathbf{R}$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}\, dt \text{ a.e.}$$

4. Let Γ be the semi-circular arc of radius R and centered at 0 lying above the x-axis.

(a) (3 pts) If $|F(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants, prove that

$$\lim_{R\to\infty} \int_\Gamma F(z)dz = 0.$$

(b) (3 pts) Show that for a large enough R and $z = Re^{i\theta}$,

$$|f(z)| \le \frac{2}{R^6}$$

if

$$f(z) = \frac{1}{z^6+1}.$$

(c) (4 pts) Evaluate

$$\int_0^\infty \frac{dx}{x^6+1}.$$

5. Suppose that the only singularities of a function $f(z)$ are the simple poles $a_1, a_2, \cdots$ arranged in order of increasing absolute value. Let the residue of $f(z)$ at a_i be b_i. Let C_N be circles of radius R_N which do not pass through any poles and on which $|f(z)| < M$, where M is independent of N and $R_N \to \infty$ as $N \to \infty$.

(a) (3 pts) Suppose $f(z)$ is analytic at $z = \zeta$. Show that

$$\frac{1}{2\pi i}\int_{C_N} \frac{f(z)}{z-\zeta}dz = f(\zeta) + \sum_n \frac{b_n}{a_n - \zeta},$$

where the last summation is taken over all poles inside circle C_N of radius R_N.

(b) (2 pts) Suppose $f(z)$ is analytic at $z = 0$. Using the above, show that

$$f(\zeta) - f(0) + \sum_n b_n\left(\frac{1}{a_n-\zeta} - \frac{1}{a_n}\right) = \frac{\zeta}{2\pi i}\int_{C_N} \frac{f(z)}{z(z-\zeta)}dz.$$

(c) (4 pts) Show that

$$\lim_{N\to\infty} \int_{C_N} \frac{f(z)}{z(z-\zeta)}dz = 0.$$

(d) (1 pt) Deduce that

$$f(\zeta) = f(0) + \sum_n b_n\left(\frac{1}{\zeta - a_n} + \frac{1}{a_n}\right).$$

Write your answers in the answer sheets provided. Give full explanation with clear statements of any theorem you use. Use no books or notes in this exam. Attempt all problems. You have 3 hours.

1. (a) (8 points) What is the number of integer solutions of the equation
$$x_1 + x_2 + x_3 = 5$$
that satisfy $1 \le x_1 \le 4$, $-2 \le x_2 \le 2$, $0 \le x_3 \le 5$?

   (b) (9 points) Determine the number of non-equivalent colorings of the corners of a regular 5-gon with colors red, white, and blue in which two corners are colored red, two are colored white, and one is colored blue.

2. (a) (8 points) Show that every sequence $a_1, a_2, \ldots, a_{n^2+1}$ of $n^2 + 1$ distinct real numbers contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

   (b) (9 points) Using exponential generating function, show that the number h_n of n digit numbers with each digit odd, where the digits 1 and 3 occur an even number of times, satisfies the formula
$$h_n = \frac{5^n + 2 \times 3^n + 1}{4}, \quad (n \ge 0).$$

3. (a) (8 points) Let $G = \{A \cup B, E\}$ be a bipartite graph and
$$\delta = \max_{X \subseteq A}\{|X| - |\Gamma_G(X)|\},$$
where $\Gamma_G(X)$ is the set of neighbours of X. Prove that minimum vertex cover size $\tau(G) = |A| - \delta$.

   (b) (8 points) If G is a simple graph, then prove that $k(G) \le k'(G) \le \delta(G)$, where $k(G)$ denotes the vertex connectivity, and $k'(G)$ denotes the edge connectivity of G.

4. (8 points) Suppose you are given an array with n numbers. Describe an algorithm to determine if the array has any duplicates. How long will your algorithm take in the worst case?

5. (8 points) Given an undirected graph, what algorithm will you use to determine if it has a cycle? How long will it take if the graph has n vertices and m edges, and the graph is stored in the memory as adjacency lists?

6. (9 points) Suppose an array $A[1, \ldots, n]$ consists of n real numbers such that $A[i] \le A[2i]$ and $A[i] \le A[2i + 1]$ (whenever the indices are in the range $\{1, 2, \ldots, n\}$). Note that this is how elements are ordered in a standard representation of heap using an array. Suppose, you are now given a number α. What algorithm will you use to print all elements in A that are at most α. If there are k such elements in A, then your algorithm should finish in time $O(k)$.

7. Suppose $G = (U, V, E, w)$ is a complete weighted bipartite graph, where each edge $e = \{u, v\}$ has a real weight $w(e)$ (which can be zero, positive or negative). For a matching M in G, let its weight be $w(M) = \sum_{e \in M} w(e)$. Let $\mathcal{M}_k$ be the set of matchings in G with k edges, and let M_k be a matching in $\mathcal{M}_k$ with maximum weight. Let H_k be the directed bipartite

graph with the same vertex sets and edge set as G, where edges of M_k are directed from V to U and all other edges directed from U to V. For an edge e in H_k, let $w(e)$ be its weight in the original graph G, when e did not have any direction; similarly, for a matching M in H_k, let $w(M) = \sum_{e \in M} w(e)$. Now, let the weight function w_k be defined by: $w_k(e) = -w(e)$ if $e \in M_k$, and $w_k(e) = w(e)$ otherwise.

(a) (5 points) Construct an example where $w(M_4)$ is positive but $w(M_3) > w(M_4)$.

(b) (5 points) For a path p in H_k, let its weight be $w_k(p) = \sum_{e \in p} w_k(e)$. Let p be a path in H_k from an unmatched vertex in U to an unmatched vertex in V. Let $M' = M_k \oplus p$ be the matching $(M_k \setminus p) \cup (p \setminus M_k)$ obtained by taking the symmetric difference of M_k and p. How many edges does M' have? What is the relationship between $w(M_k)$, $w_k(p)$ and $w(M')$?

(c) (6 points) Show that H_k has no positive weight cycle under weight function w_k, where the weight of the cycle C is

$$w_k(C) = \sum_{e \in C} w_k(e).$$

(d) (9 points) Recall that H_k might have both positive weight edges but no positive weight directed cycles under weight function w_k. Let U_0 be the unmatched vertices in U and V_0 be the unmatched vertices in V. Describe an algorithm to efficiently find a maximum weight path from U_0 to V_0 with respect to w_k. State the worst case running time of your algorithm in terms of n, the number vertices.

COMPREHENSIVE EXAMINATION, DECEMBER 2013
TOPOLOGY

Problem 1. (i) Suppose $p : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ and $q : S^1 \to \mathbb{R}^2 \setminus \{(0,0)\}$ be two loops on the plane $\mathbb{R}^2 \setminus \{(0,0)\}$. Prove that if $|p(t) - q(t)| < |p(t)|$ for all $t \in S^1$, p and q are homotopic to each other. (5 points.)

(ii) Find the fundamental group, $\pi_1(S^3 \setminus S^1)$ where S^3 is a 3 dimensional sphere and $S^1 = \{(x, y, z, t) \in S^3 \mid z = t = 0\}$. (5 points.)

(iii) Prove that a map from $S^2 \to S^1 \times S^1$ is null homotopic. What is $\pi_2(S^1 \times S^1)$? (5 points.)

(iv) Prove that the cone of the identity map $X \to X$ is contractible. (2 points.)

Problem 2. (i) Find the homology groups of $\mathbb{R}^3 \setminus S^1$ using Mayer Vietoris. Here $S^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1;\ z = 0\}$. (10 points.)

(ii) Find the fundamental group of $\mathbb{R}^3 \setminus S^1$ using Van Kampen's theorem. (10 points.)

Problem 3. (i) Find the homology groups of $S^2 \times S^3$. (5 points.)

(ii) Find the homology groups of $S^2 \vee S^3 \vee S^5$. (10 points.)

(iii) Find the relative homology groups $H_i(S^3, S^1)$ where S^3 and S^1 are as in problem 1(ii). (5 points.)

Problem 4. (i) Find the cohomology groups of $S^2 \times S^3$. (5 points.)

(ii) Find

(a) a 2 form on S^2 and

(b) a 3 form on S^3

which are closed but not exact. (6 points)

(iii) Find a closed 5-form on $S^2 \times S^3$ which is not exact. (3 points.)

Problem 5. (i) Find the cohomology groups of $S^2 \vee S^3 \vee S^5$. (5 points.)

(ii) Are $S^2 \times S^3$ and $S^2 \vee S^3 \vee S^5$ homeomorphic? Prove your answer using algebraic topological invariants. (*Hint : Assuming the contrary, look at the complement of a suitable point and its image.*) (10 points.)

Problem 6. Let R be a commutative ring with 1. Let X and Y be two topological spaces and $X \sqcup Y$ be their disjoint union. Prove that

$$H^*(X \sqcup Y, R) \cong H^*(X, R) \times H^*(Y, R)$$

where the product structure on the right is $(\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) = (\alpha_1 \cup \alpha_2, \beta_1 \cup \beta_2)$. Using this compute $H^*(X \vee Y, R)$. (*Hint :* $H^i(X \vee Y) \xrightarrow[q^*]{\sim} H^i(X \sqcup Y)$ *for* $i \geq 1$, *where* $q : X \sqcup Y \to X \vee Y$ *is the quotient map.*) (14 points.)